

7-1

Ratios and Proportions

Common Core State Standards

Prepares for G-SRT.B.5 Use . . . similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

MP 1, MP 3, MP 4, MP 6, MP 7

Objective To write ratios and solve proportions



The year the team had the most wins is not necessarily the year in which it had the best record.



Getting Ready!

The table at the right gives the wins and losses of a baseball team. In which year(s) did the team have the best record? Explain.

Year	Wins	Losses
1890	60	24
1930	110	44
1970	110	52
2010	108	54



MATHEMATICAL PRACTICES

In the Solve It, you compared two quantities for four years.

Essential Understanding You can write a *ratio* to compare two quantities.

A **ratio** is a comparison of two quantities by division. You can write the ratio of two numbers a and b , where $b \neq 0$, in three ways: $\frac{a}{b}$, $a : b$, and a to b . You usually express a and b in the same unit and write the ratio in simplest form.



Lesson Vocabulary

- ratio
- extended ratio
- proportion
- extremes
- means
- Cross Products Property

Think

How can you write the heights using the same unit?

You can convert the height of the Senator to inches or the height of the bonsai tree to feet.



Problem 1 Writing a Ratio

Bonsai Trees The bonsai bald cypress tree is a small version of a full-size tree. A Florida bald cypress tree called the Senator stands 118 ft tall. What is the ratio of the height of the bonsai to the height of the Senator?

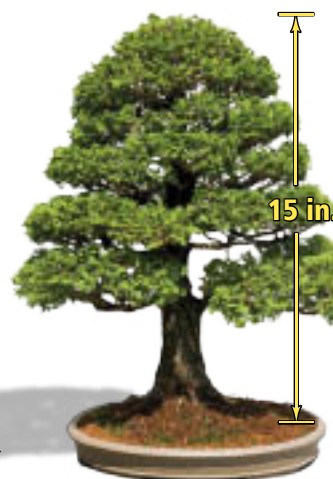
Express both heights in the same unit. To convert 118 ft to inches, multiply by the conversion factor $\frac{12 \text{ in.}}{1 \text{ ft}}$.

$$118 \text{ ft} = \frac{118 \cancel{\text{ft}}}{1} \cdot \frac{12 \text{ in.}}{1 \cancel{\text{ft}}} = (118 \cdot 12) \text{ in.} = 1416 \text{ in.}$$

Write the ratio as a fraction in simplest form.

$$\frac{\text{height of bonsai}}{\text{height of Senator}} \rightarrow \frac{15 \text{ in.}}{118 \text{ ft}} = \frac{15 \text{ in.}}{1416 \text{ in.}} = \frac{(3 \cdot 5) \cancel{\text{in.}}}{(3 \cdot 472) \cancel{\text{in.}}} = \frac{5}{472}$$

The ratio of the height of the bonsai to the height of the Senator is $\frac{5}{472}$ or 5 : 472.



Got It? 1. A bonsai tree is 18 in. wide and stands 2 ft tall. What is the ratio of the width of the bonsai to its height?



Problem 2 Dividing a Quantity Into a Given Ratio

Fundraising Members of the school band are buying pots of tulips and pots of daffodils to sell at their fundraiser. They plan to buy 120 pots of flowers. The ratio $\frac{\text{number of tulip pots}}{\text{number of daffodil pots}}$ will be $\frac{2}{3}$. How many pots of each type of flower should they buy?

Plan

How do you write expressions for the numbers of pots? Multiply the numerator 2 and the denominator 3 by the factor x . $\frac{2x}{3x} = \frac{2}{3}$.

Think

If the ratio $\frac{\text{number of tulip pots}}{\text{number of daffodil pots}}$ is $\frac{2}{3}$, it must be in the form $\frac{2x}{3x}$.

The total number of flower pots is 120. Use this fact to write an equation. Then solve for x .

Substitute 24 for x in the expressions for the numbers of pots.

Write the answer in words.

Write

Let $2x$ = the number of tulip pots.

Let $3x$ = the number of daffodil pots.

$$2x + 3x = 120$$

$$5x = 120$$

$$x = 24$$

$$2x = 2(24) = 48$$

$$3x = 3(24) = 72$$

The band members should buy 48 tulip pots and 72 daffodil pots.



Got It? 2. The measures of two supplementary angles are in the ratio 1 : 4. What are the measures of the angles?

An **extended ratio** compares three (or more) numbers. In the extended ratio $a : b : c$, the ratio of the first two numbers is $a : b$, the ratio of the last two numbers is $b : c$, and the ratio of the first and last numbers is $a : c$.



Problem 3 Using an Extended Ratio

The lengths of the sides of a triangle are in the extended ratio 3 : 5 : 6. The perimeter of the triangle is 98 in. What is the length of the longest side?

Sketch the triangle. Use the extended ratio to label the sides with expressions for their lengths.

$$3x + 5x + 6x = 98$$

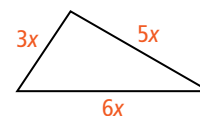
$$14x = 98$$

$$x = 7$$

The perimeter is 98 in.

Simplify.

Divide each side by 14.



The expression that represents the length of the longest side is $6x$. $6(7) = 42$, so the length of the longest side is 42 in.



Got It? 3. The lengths of the sides of a triangle are in the extended ratio 4 : 7 : 9. The perimeter is 60 cm. What are the lengths of the sides?

Think

How do you use the solution of the equation to answer the question? Substitute the value for x in the expression for the length of the longest side.

Essential Understanding If two ratios are equivalent, you can write an equation stating that the ratios are equal. If the equation contains a variable, you can solve the equation to find the value of the variable.

An equation that states that two ratios are equal is called a **proportion**. The first and last numbers in a proportion are the **extremes**. The middle two numbers are the **means**.



Key Concept Cross Products Property

Words

In a proportion, the product of the **extremes** equals the product of the **means**.

Symbols

If $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$ and $d \neq 0$, then $ad = bc$.

Example

$$\begin{aligned} \frac{2}{3} &= \frac{4}{6} \\ 2 \cdot 6 &= 3 \cdot 4 \\ 12 &= 12 \end{aligned}$$

Here's Why It Works Begin with $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$ and $d \neq 0$.

$$bd \cdot \frac{a}{b} = \frac{c}{d} \cdot bd$$

Multiply each side of the proportion by bd .

$$\frac{bd}{1} \cdot \frac{a}{b} = \frac{c}{d} \cdot \frac{bd}{1}$$

Divide the common factors.

$$ad = bc$$

Simplify.



Problem 4 Solving a Proportion

Algebra What is the solution of each proportion?

A $\frac{6}{x} = \frac{5}{4}$

$$6(4) = 5x$$

$$24 = 5x$$

$$x = \frac{24}{5}$$

Cross Products Property

Simplify.

Solve for the variable.

The solution is $\frac{24}{5}$ or 4.8.

B $\frac{y+4}{9} = \frac{y}{3}$

$$3(y+4) = 9y$$

$$3y + 12 = 9y$$

$$12 = 6y$$

$$y = 2$$

The solution is 2.

Think

Does the solution check?

$$\begin{aligned} \frac{6}{\frac{24}{5}} &\stackrel{?}{=} \frac{5}{4} \\ 6 \cdot 4 &\stackrel{?}{=} \frac{24}{5} \cdot 5 \\ 24 &= 24 \checkmark \end{aligned}$$



Got It? 4. What is the solution of each proportion?

a. $\frac{9}{2} = \frac{a}{14}$

b. $\frac{15}{m+1} = \frac{3}{m}$

Using the Properties of Equality, you can rewrite proportions in equivalent forms.

Take note

Key Concept Properties of Proportions

a , b , c , and d do not equal zero.

Property

(1) $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{b}{a} = \frac{d}{c}$.

(2) $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{a}{c} = \frac{b}{d}$.

(3) $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{a+b}{b} = \frac{c+d}{d}$.

How to Apply It

Write the reciprocal of each ratio.

$$\left(\frac{2}{3} = \frac{4}{6}\right) \text{ becomes } \frac{3}{2} = \frac{6}{4}.$$

Switch the means.

$$\frac{2}{3} = \frac{4}{6} \text{ becomes } \frac{2}{4} = \frac{3}{6}.$$

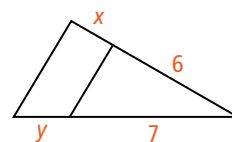
In each ratio, add the denominator to the numerator.

$$\frac{2}{3} = \frac{4}{6} \text{ becomes } \frac{2+3}{3} = \frac{4+6}{6}.$$



Problem 5 Writing Equivalent Proportions

In the diagram, $\frac{x}{6} = \frac{y}{7}$. What ratio completes the equivalent proportion $\frac{x}{y} = \frac{\square}{\square}$? Justify your answer.



Plan

How do you decide which property of proportions applies? Look at how the positions of the known parts of the incomplete proportion relate to their positions in the original proportion.

Method 1

$$\frac{x}{6} = \frac{y}{7}$$

$$\frac{x}{y} = \frac{6}{7} \quad \text{Property of Proportions (2)}$$

Method 2

$$\frac{x}{6} = \frac{y}{7}$$

$$7x = 6y \quad \text{Cross Products Property}$$

$$\frac{7x}{7y} = \frac{6y}{7y} \quad \text{To solve for } \frac{x}{y}, \text{ divide each side by } 7y.$$

$$\frac{x}{y} = \frac{6}{7} \quad \text{Simplify.}$$

The ratio that completes the proportion is $\frac{6}{7}$.



Got It? 5. For parts (a) and (b), use the proportion $\frac{x}{6} = \frac{y}{7}$. What ratio completes the equivalent proportion? Justify your answer.

a. $\frac{6}{x} = \frac{\square}{\square}$

b. $\frac{\square}{\square} = \frac{y+7}{7}$

c. **Reasoning** Explain why $\frac{6}{x-6} = \frac{7}{y-7}$ is an equivalent proportion to $\frac{x}{6} = \frac{y}{7}$.



Lesson Check

Do you know HOW?

- To the nearest millimeter, a cell phone is 84 mm long and 46 mm wide. What is the ratio of the width to the length?
- Two angle measures are in the ratio 5 : 9. Write expressions for the two angle measures in terms of the variable x .
- What is the solution of the proportion $\frac{20}{z} = \frac{5}{3}$?
- For $\frac{a}{7} = \frac{13}{b}$ complete each equivalent proportion.
 - $\frac{a}{\square} = \frac{7}{\square}$
 - $\frac{a-7}{7} = \frac{\square}{\square}$
 - $\frac{7}{a} = \frac{\square}{\square}$

Do you UNDERSTAND?



- Vocabulary** What is the difference between a ratio and a proportion?
- Open-Ended** The lengths of the sides of a triangle are in the extended ratio 3 : 6 : 7. What are two possible sets of side lengths, in inches, for the triangle?
- Error Analysis** What is the error in the solution of the proportion shown at the right?
- What is a proportion that has means 6 and 18 and extremes 9 and 12?

$$\begin{array}{rcl} \frac{7}{3} & = & \frac{4}{x} \\ 28 & = & 3x \\ \frac{28}{3} & = & x \end{array}$$



Practice and Problem-Solving Exercises



A Practice

Write the ratio of the first measurement to the second measurement.

See Problem 1.

- length of a tennis racket: 2 ft 4 in.
length of a table tennis paddle: 10 in.
- height of a table tennis net: 6 in.
height of a tennis net: 3 ft
- diameter of a table tennis ball: 40 mm
diameter of a tennis ball: 6.8 cm
- length of a tennis court: 26 yd
length of a table tennis table: 9 ft
- Baseball** A baseball team played 154 regular season games. The ratio of the number of games they won to the number of games they lost was $\frac{5}{2}$. How many games did they win? How many games did they lose?
- The measures of two supplementary angles are in the ratio 5 : 7. What is the measure of the larger angle?
- The lengths of the sides of a triangle are in the extended ratio 6 : 7 : 9. The perimeter of the triangle is 88 cm. What are the lengths of the sides?
- The measures of the angles of a triangle are in the extended ratio 4 : 3 : 2. What is the measure of the largest angle?

See Problem 2.

See Problem 3.

Algebra Solve each proportion.

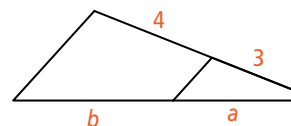
See Problem 4.

- $\frac{1}{3} = \frac{x}{12}$
- $\frac{9}{5} = \frac{3}{x}$
- $\frac{4}{x} = \frac{5}{9}$
- $\frac{y}{10} = \frac{15}{25}$
- $\frac{9}{24} = \frac{12}{n}$
- $\frac{11}{14} = \frac{b}{21}$
- $\frac{3}{5} = \frac{6}{x+3}$
- $\frac{y+7}{9} = \frac{8}{5}$
- $\frac{5}{x-3} = \frac{10}{x}$
- $\frac{n+4}{8} = \frac{n}{4}$

In the diagram, $\frac{a}{b} = \frac{3}{4}$. Complete each statement. Justify your answer.

← See Problem 5.

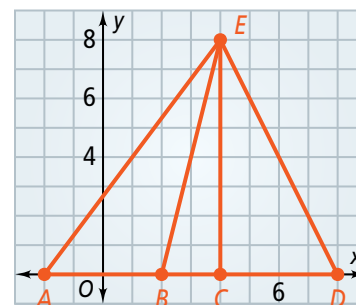
27. $\frac{b}{a} = \frac{\square}{\square}$ 28. $4a = \square$ 29. $\frac{\square}{\square} = \frac{b}{4}$
 30. $\frac{\square}{\square} = \frac{7}{4}$ 31. $\frac{a+b}{b} = \frac{\square}{\square}$ 32. $\frac{b}{\square} = \frac{4}{\square}$



B Apply

Coordinate Geometry Use the graph. Write each ratio in simplest form.

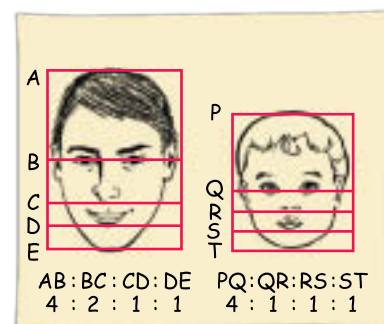
33. $\frac{AC}{BD}$ 34. $\frac{AE}{EC}$ 35. slope of \overline{EB} 36. slope of \overline{ED}



- © 37. **Think About a Plan** The area of a rectangle is 150 in.^2 . The ratio of the length to the width is $3 : 2$. Find the length and the width.
- What is the formula for the area of a rectangle?
 - How can you use the given ratio to write expressions for the length and width?

Art To draw a face, you can sketch the head as an oval and then lightly draw horizontal lines to help locate the eyes, nose, and mouth. You can use the extended ratios shown in the diagrams to help you place the lines for an adult's face or for a baby's face.

38. If $AE = 72 \text{ cm}$ in the diagram, find AB , BC , CD , and DE .
 39. You draw a baby's head as an oval that is 21 in. from top to bottom.
 a. How far from the top should you place the line for the eyes?
 b. Suppose you decide to make the head an adult's head. How far up should you move the line for the eyes?



Algebra Solve each proportion.

40. $\frac{1}{7y-5} = \frac{2}{9y}$ 41. $\frac{4a+1}{7} = \frac{2a}{3}$ 42. $\frac{5}{x+2} = \frac{3}{x+1}$ 43. $\frac{2b-1}{4} = \frac{b-2}{12}$

44. The ratio of the length to the width of a rectangle is $9 : 4$. The width of the rectangle is 52 mm . Write and solve a proportion to find the length.
- © 45. **Open-Ended** Draw a quadrilateral that satisfies this condition: The measures of the consecutive angles are in the extended ratio $4 : 5 : 4 : 7$.
- © 46. **Reasoning** The means of a proportion are 4 and 15 . List all possible pairs of positive integers that could be the extremes of the proportion.
- © 47. **Writing** Describe how to use the Cross Products Property to determine whether $\frac{10}{26} = \frac{16}{42}$ is a true proportion.
- © 48. **Reasoning** Explain how to use two different properties of proportions to change the proportion $\frac{3}{4} = \frac{12}{16}$ into the proportion $\frac{12}{3} = \frac{16}{4}$.

Complete each statement. Justify your answer.

49. If $4m = 9n$, then $\frac{m}{n} = \frac{\square}{\square}$.

50. If $\frac{30}{t} = \frac{18}{r}$, then $\frac{t}{r} = \frac{\square}{\square}$.

51. If $\frac{a+5}{5} = \frac{b+2}{2}$, then $\frac{a}{5} = \frac{\square}{\square}$.

52. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{c+d} = \frac{\square}{\square}$.

53. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = \frac{\square}{\square}$.

54. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+2b}{b} = \frac{\square}{\square}$.



Challenge

Algebra Use properties of equality to justify each property of proportions.

55. $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{b}{a} = \frac{d}{c}$.

56. $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{a}{c} = \frac{b}{d}$.

57. $\frac{a}{b} = \frac{c}{d}$ is equivalent to $\frac{a+b}{b} = \frac{c+d}{d}$.

Algebra Solve each proportion for the variable(s).

58. $\frac{x-3}{3} = \frac{2}{x+2}$

59. $\frac{3-4x}{1+5x} = \frac{1}{2+3x}$

60. $\frac{x}{6} = \frac{x+10}{18} = \frac{4x}{y}$



Apply What You've Learned



**MATHEMATICAL
PRACTICES**

MP 1

Look back at the information on page 431 about the dimensions of Lillian's graphing calculator screen. Select all of the following that are true. Explain your reasoning.

- A. The perimeter of the calculator screen is 10 cm.
- B. The ratio of the width of the screen to the height of the screen is 3 : 2.
- C. The ratio of the width of the screen to the height of the screen is 15 : 9.
- D. The ratio of the width of the screen to the height of the screen is the same whether you measure the dimensions in centimeters or millimeters.
- E. A rectangle with a width of 2 ft and a height of 16 in. has the same ratio of width to height as Lillian's calculator screen.
- F. An extended ratio that compares the dimensions of the four sides of Lillian's calculator screen is 3 : 2 : 3 : 2.
- G. The extended ratio that compares the dimensions of the four sides of Lillian's calculator screen is 2 : 1 : 2 : 1.